

Algebraic quantum hypergroups imbedded in algebraic quantum groups

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Abstract

We show that for any $*$ -algebraic quantum group, the space of invariants for the square of the antipode carries the structure of a $*$ -algebraic quantum hypergroup.

Introduction

In [20], A. Van Daele introduced the notion of a $*$ -algebraic quantum group. These objects united and generalized compact and discrete quantum groups in an entirely algebraic framework, and they formed a major motivation for the general theory of locally compact quantum groups as developed in [9]. In [5], the notion of a $*$ -algebraic quantum group was generalized to that of a $*$ -algebraic quantum *hypergroup*. The main difference with a $*$ -algebraic quantum group is that the comultiplication Δ is no longer required to be an algebra morphism, although it still commutes with the $*$ -structure. However, the comultiplication and multiplication are now tied together in a weak way, by the existence of an anti-automorphism, which is some kind of *antipode*, satisfying a form of ‘strong left invariance’ with respect to a ‘left Haar functional’.

The main construction of *classical* hypergroups consists of considering double coset spaces for a group-subgroup pair. For some partial generalizations of this construction to the quantum setting, see e.g. [17], [1], [8] and [14]. In this paper, we will consider a general construction method of $*$ -algebraic quantum hypergroups from $*$ -algebraic quantum groups, which is *only* interesting in the quantum setting. Namely, we show that the space of fixed points of the antipode squared has a natural $*$ -algebraic quantum hypergroup structure.

The layout of this paper is as follows. *In the first section* we show that for every $*$ -algebraic quantum group, there exists a certain compact group which acts on it by comultiplication-preserving $*$ -automorphisms. We can then construct a crossed product, which can be endowed naturally with a $*$ -algebraic quantum group structure. In fact, this is a (very special) example of the bicrossed product of $*$ -algebraic quantum groups. Using a certain distinguished group-like element in this new $*$ -algebraic quantum group, we can then construct a $*$ -algebraic quantum hypergroup within A . In particular, this will allow us to show that the set of fixed points of the antipode squared carries a

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natural structure of $*$ -algebraic quantum hypergroup. We note that this construction could also have been carried out in a more straightforward fashion, but it seemed nice to use this more elaborate procedure as to provide a link with the work in [6].

In the second section, we will see how this construction, when applied to the dual $*$ -algebraic quantum group, is naturally the dual of the $*$ -algebraic quantum hypergroup created from the original $*$ -algebraic quantum group.

In the third section, we apply these construction methods to Woronowicz' twisted $SU_q(2)$ and its dual $\widehat{SU}_q(2)$. In the case of $SU_q(2)$, we provide some concrete formulas. We note that the treatment of this example does not really need any of the previous abstract theory, but at least it shows one can get interesting phenomena by this construction.

Conventions and remarks on notation

By ι we always mean the identity map. We use \otimes for the tensor product between vector spaces over \mathbb{C} , and, in the third section, $\bar{\otimes}$ for the spatial tensor product between von Neumann algebras. The dual of a vector space V is denoted as V° , since we use the symbol $*$ for $*$ -operations.

In this paper, we will need to work with non-unital $*$ -algebras. When X is a non-degenerate $*$ -algebra, we denote by $M(X)$ the unital $*$ -algebra of multipliers of X (see e.g. the appendix of [18]). When $p \in X$ is a (self-adjoint) projection, we will identify $pM(X)p$ with $M(pXp)$ as $*$ -algebras in the natural way.

For the notion of a $*$ -algebraic quantum group, we refer the reader to [20]. The concept of a $*$ -algebraic quantum hypergroup was introduced in [5]. We recall that the main difference between these concepts is that in the latter structure, the comultiplication is not assumed to be a homomorphism.

1 A quantum hypergroup structure on $\text{Fix}(S^2)$

We begin with recalling the following result from [3].

Proposition 1.1. *Let (A, Δ) be a $*$ -algebraic quantum group with antipode S . Then there exists a basis of A consisting of eigenvectors for the linear map S^2 .*

For the rest of this section, (A, Δ) will then be a fixed $*$ -algebraic quantum group with antipode S .

Definition 1.2. *We denote by $T \subseteq \mathbb{R}_0^+$ the set of eigenvalues of S^2 , considered as a linear map $A \rightarrow A$.*

We denote by G the subgroup of (\mathbb{R}_0^+, \cdot) generated by T .

For $r \in T$, we denote by $V_r \subseteq A$ its associated space of eigenvectors.

We denote $C = V_1 = \text{Fix}(S^2)$, the space of fixed elements for S^2 .

Now let $K(G)$ be the $*$ -algebra of functions with finite support on G . It can be made into a $*$ -algebraic quantum group by endowing it with the comultiplication Ψ determined by

$$\Psi(\delta_r) = \sum_{s \in G} \delta_{rs^{-1}} \otimes \delta_s,$$

where δ_r denotes the Dirac function at the element $r \in G$ (see for example the introduction of [20]). We will now construct a new $*$ -algebraic quantum group, starting from (A, Δ) and $(K(G), \Psi)$. The construction of this algebra is inspired by a result of Connes concerning quasi-periodic weights on von Neumann algebras, and by a remark in [10]. Namely, one can take the dual compact abelian group \widehat{G} , and let it act on A by an action α satisfying the formula

$$\alpha_\chi(a) = \langle \chi, r \rangle a \quad \chi \in \widehat{G}, r \in T, a \in V_r.$$

Then each α_χ can be shown to be an automorphism of the $*$ -algebraic quantum group, and by this one can construct a quantum group structure on the crossed product. In the following, we will carry out this construction in detail, using however smash products instead of crossed products (although these are really equivalent concepts in this context).

We will first need to prove a couple of lemmas.

Lemma 1.3. *There exists a unique unital left $K(G)$ -module $*$ -algebra structure on A such that*

$$f \triangleright a = f(r)a \quad r \in T, a \in V_r, f \in K(G).$$

We recall that a unital left module $*$ -algebra structure is a unital left module structure $(K(G) \triangleright A = A)$, interacting with the $*$ -algebra structure in the following way:

$$x \triangleright (yz) = (x_{(1)} \triangleright y) \cdot (x_{(2)} \triangleright z),$$

$$x \triangleright (y^*) = ((S(x)^*) \triangleright y)^*.$$

Proof. By Proposition 1.1, it is clear that the formula in the statement of the Lemma can be extended linearly to a unital left $K(G)$ -module structure on A . Since S^2 is an automorphism, we also have that if $r, s \in T$, then $V_r \cdot V_s \subseteq V_{rs}$, from which the module algebra property follows. Finally, since $S(a)^* = S^{-1}(a^*)$ for $a \in A$, we have $V_r^* = V_{r^{-1}}$, from which the module $*$ -property follows. \square

For the following lemma, we make some remarks. Let H be a $*$ -algebraic quantum group, X a non-degenerate $*$ -algebra, and suppose that we have a unital left module $*$ -algebra structure \triangleright of H on X . Then we first remark that $X \otimes X$ becomes a unital left $H \otimes H$ -module $*$ -algebra in the natural way. A second remark is that \triangleright can be extended to a H -module structure on $M(X)$: if $x \in M(X)$ and $h \in H$, we make sense of $h \triangleright x$ as a multiplier by the following formulas: if $y \in X$ and e is an element in H such that $e \triangleright y = y$ (which exists by the unitality of the left H -module X and the existence of local units in H), we define

$$(h \triangleright x) \cdot y = h_{(1)} \triangleright (x \cdot ((S(S^{-1}(e)h_{(2)})) \triangleright y)), \quad y \in X,$$

$$y \cdot (h \triangleright x) = h_{(2)} \triangleright ((S^{-1}(S(e)h_{(1)}) \triangleright y) \cdot x), \quad y \in X.$$

Finally, for $h \in M(H)$ and $x \in M(X)$, we can then also make sense of $h \triangleright x$ as a map from H to $M(X)$, sending $g \in H$ to $(gh) \triangleright x$. See the discussion in section 3 and appendix B of [6] for some more information on the well-definedness of these constructions.

We can now make sense of the identity in the following lemma.

Lemma 1.4. *For $a \in A$ and $f \in K(G)$, we have*

$$\Delta(f \triangleright a) = \Psi(f) \triangleright \Delta(a).$$

Proof. Choose $u \in G$, $r \in T$ and $a \in V_r$. It is sufficient to prove the above identity for $f = \delta_u$. Choose also $s, t \in G$. Then, by the above discussion, it is sufficient to prove the above identity when applying $\delta_s \otimes \delta_t$ to the left of it, which leads to the identity

$$\delta_{u,r}(\delta_s \otimes \delta_t) \triangleright \Delta(a) = \delta_{st,u}(\delta_s \otimes \delta_t) \triangleright \Delta(a).$$

Now choose yet another $w \in T$, and put $b \in V_w$. Then multiplying the previous expressions to the right with $(1 \otimes b)$, and again using the remarks before the Lemma, we see that it becomes sufficient to prove that

$$\delta_{st,u}(\delta_s \otimes \delta_{tw}) \triangleright (\Delta(a)(1 \otimes b)) = \delta_{u,r}(\delta_s \otimes \delta_{tw}) \triangleright (\Delta(a)(1 \otimes b)).$$

Write $\Delta(a)(1 \otimes b)$ as a finite sum of elements $p_i \otimes q_i$, where the q_i are linearly independent elements such that $q_i \in V_{r_i}$ for some $r_i \in T$. Choose $\omega_i \in A^\circ$ dual to the q_i , so that $\omega_j(q_i) = \delta_{ij}$. Then

$$p_i = (\iota \otimes \omega)(\Delta(a)(1 \otimes b)),$$

and since S^2 is a Δ -preserving automorphism, we find

$$\begin{aligned} S^2(p_i) &= (\iota \otimes \omega)((S^2 \otimes \iota)(\Delta(a)(1 \otimes b))) \\ &= (\iota \otimes (\omega \circ S^{-2}))(\Delta(S^2(a))(1 \otimes S^2(b))) \\ &= rw \sum \omega(S^{-2}(q_i))p_i \\ &= rwr_i^{-1}p_i, \end{aligned}$$

and hence $p_i \in V_{rwr_i^{-1}}$. But this means that when $(\delta_s \otimes \delta_{tw}) \triangleright (\Delta(a)(1 \otimes b))$ is not zero, necessarily $st = r$. This concludes the proof. \square

The proofs of the following two propositions will be given after the statement of the second one.

Proposition 1.5. *Let B be the vectorspace $A \otimes k(G)$. Then B can be made into a non-degenerate $*$ -algebra by endowing it with the multiplication*

$$(a \otimes \delta_r) \cdot (b \otimes \delta_s) = \delta_{r,ts} ab \otimes \delta_s, \quad r, s \in G, t \in T, a \in A, b \in V_t$$

and the $*$ -operation

$$(a \otimes \delta_r)^* = a^* \otimes \delta_{sr}, \quad r \in G, s \in T, a \in V_s.$$

Remark: We can then imbed the $*$ -algebra A into $M(B)$ by sending a to $a \otimes 1$, which is easily interpreted as a multiplier of B . Similarly, $K(G)$ can be imbedded into $M(B)$ by sending f to $1 \otimes f$. We will then identify A and $K(G)$ with their images inside $M(B)$, and we then have $B = A \cdot K(G) = K(G) \cdot A$. The following commutation relations are easily verified to hold in $M(B)$:

$$\delta_r a = a \delta_{s^{-1}r}, \quad r \in G, s \in T, a \in V_s.$$

Proposition 1.6. *There exists a unique $*$ -algebraic quantum group structure (B, Δ_B) on B , such that for all $a, b \in A$ and $f, g \in K(G)$, we have*

$$(1 \otimes a)\Delta_B(bf)(1 \otimes g) = (1 \otimes a)\Delta(b) \cdot \Psi(f)(1 \otimes g).$$

Proof. We give A the unital left $*$ -module algebra structure \triangleright by $K(G)$ as before, and consider $K(G)$ as a trivial right A -comodule $*$ -coalgebra by the comodule map $A \rightarrow A \otimes K(G) : a \rightarrow a \otimes 1$. We then check that these two structures satisfy the compatibility conditions of Theorem 3.16 of [6], from which the result will follow from (the dual version of) Theorem 4.3 of [6].

Now the conditions stated there, written out in Sweedler notation, say that the following identities should hold for all $f, g \in K(G)$ and $a \in A$:

$$\begin{aligned} \sum f_{(1)}g_{(1)} \otimes (f_{(2)}g_{(2)}) \triangleright a &= \sum f_{(1)}g_{(1)} \otimes f_{(2)} \triangleright (g_{(2)} \triangleright a), \\ \sum f_{(1)} \otimes \Delta(d_{(2)} \triangleright a) &= \sum f_{(1)} \otimes f_{(2)} \triangleright a_{(1)} \otimes f_{(3)} \triangleright a_{(2)}, \\ \sum f_{(1)} \otimes f_{(2)} \triangleright a &= \sum f_{(2)} \otimes f_{(1)} \triangleright a. \end{aligned}$$

Then the first identity follows by the module property, the second by Lemma 1.4, and the third by the cocommutativity of $K(G)$. □

Remarks:

1. In [16], a very general kind of bicrossed product construction was considered in the framework of locally compact quantum groups. Since $*$ -algebraic quantum groups are particular kinds of locally compact quantum groups (by [10]), one could also invoke the results of that paper to obtain the previous result. We have decided to stick however with purely algebraic machinery.
2. By the formulas of [6], it follows that a left invariant positive functional φ_B on B is provided by

$$\varphi_B(af) = \varphi(a) \cdot \sum_{r \in G} f(r),$$

where φ is a left invariant positive functional on A .

3. Let Q be the element $\sum_{r \in G} r\delta_r \in M(K(G)) = F(G)$, the $*$ -algebra of all functions on G . Then $Q^* = Q$ and $\Psi(Q) = Q \otimes Q$. Inside $M(B)$, we can then consider the $*$ -algebra generated by A and Q , which will be the universal $*$ -algebra generated by A and Q with the commutation relation $Qa = S^2(a)Q$. The comultiplication restricts to it as $\Delta_B(a) = \Delta(a)$ and $\Delta_B(Q) = Q \otimes Q$. So we obtain a multiplier Hopf $*$ -algebra structure on the crossed product $A \rtimes \mathbb{Z}$, where \mathbb{Z} acts on A by even powers of the antipode. It is clear that *this* construction will work for any multiplier Hopf ($*$)-algebra, but when we apply it to a $*$ -algebraic quantum group, *this* construction will (in general) not give us back a $*$ -algebraic quantum group.

The following definition is taken from [13].

Definition 1.7. *If (H, Δ) is a $*$ -algebraic quantum group, a group-like projection in H is a self-adjoint projection e in $M(H)$ satisfying $\Delta(e)(1 \otimes e) = e \otimes e$.*

Definition 1.8. *With (A, Δ) and (B, Δ_B) as before, we define p as the group-like projection δ_1 in $M(B)$, where $1 \in G \subseteq \mathbb{R}_0^+$.*

From [13], we get that pBp can naturally be turned into a $*$ -algebraic quantum hypergroup (pBp, Δ_{pBp}) . We now want to find an isomorphic copy of it, more directly in terms of (A, Δ) .

We recall that we write $C = V_1 = \text{Fix}(S^2)$.

Lemma 1.9. *We have $pBp \cong^\pi C \subseteq A$ as $*$ -algebras, for some natural $*$ -isomorphism π .*

Proof. Take an element $b \in B$. Then there exists a unique function $G \times G \rightarrow A : (r, s) \rightarrow a_{r,s}$, zero almost everywhere, such that $a_{r,s} \in V_s$ and $b = \sum_{r,s} a_{r,s} \cdot \delta_r$. We then find

$$\begin{aligned} pbp &= \sum_{r,s} a_{r,s} \cdot \delta_{s^{-1}} \cdot \delta_r \cdot \delta_1 \\ &= \sum_{r,s} \delta_{s^{-1},r} \delta_{r,1} a_{r,s} \cdot \delta_1 \\ &= a_{1,1} \delta_1. \end{aligned}$$

Hence $\pi^{-1} : a \rightarrow ap$ defines a linear bijection from $\text{Fix}(S^2)$ to pBp . It is immediately seen to be a $*$ -isomorphism. \square

Definition 1.10. *We define*

$$E : A \rightarrow A : a \rightarrow \delta_1 \triangleright a.$$

Lemma 1.11. *The map E is idempotent, commutes with the $*$ -operation and has range equal to C . Moreover, it is C -bilinear.*

Proof. The first three statements are trivial to verify. Also the bilinearity property is easily seen to hold true, since for $r \in T$, the space V_r is a C -bimodule. \square

Proposition 1.12. *Under the isomorphism $\pi : pBp \rightarrow C$, the map Δ_{pBp} gets carried into*

$$\Delta_C : C \rightarrow M(C \otimes C) : c \rightarrow (E \otimes E)(\Delta(c)), \quad c \in C.$$

Remark that $(E \otimes E)(\Delta(c))$ can be made sense of as an element of $M(C \otimes C)$ by the C -bimodularity property of E .

Proof. The comultiplication on pBp is obtained from Δ_B by cutting down with $p \otimes p$:

$$\Delta_{pBp}(x) = (p \otimes p)\Delta_B(x)(p \otimes p) \quad x \in pBp.$$

Choose now $c, d \in C$. Then

$$\begin{aligned} \Delta_{pBp}(\pi(c))(1 \otimes \pi(d)) &= (p \otimes p)\Delta_B(pc)(p \otimes p)(1 \otimes pd) \\ &= (p \otimes p)\Psi(p)\Delta(c)(1 \otimes d)(p \otimes p) \\ &= (p \otimes p)\Delta(c)(1 \otimes d)(p \otimes p). \end{aligned}$$

Hence it is sufficient to prove that for $a \in A$, the equality $pap = E(a)p$ holds in B . But $pap = p(ap)p$, and hence this identity follows from the computation in Lemma 1.9. \square

Remark: in [2], an abstract theory of special conditional expectations on the C^* -algebras of compact quantum groups is created, as to create a compact quantum hypergroup structure on the range of the conditional expectation in exactly the same way as above. One can check that when our $*$ -algebraic quantum group is of compact type, the above map E is indeed a conditional expectation satisfying these conditions. However, for arbitrary $*$ -algebraic quantum groups, it is not yet clear what precise properties are needed on E to establish that we get a $*$ -algebraic quantum hypergroup in the above way. In [4], a set of conditions is proposed, but it seems too restrictive to include the above example.

2 Considering the duals

We keep the notation as in the previous section. That is, (A, Δ) is a fixed $*$ -algebraic quantum group, and (C, Δ_C) is its associated $*$ -algebraic quantum hypergroup structure on the space of S^2 -fixed elements. We will further denote by $(\widehat{A}, \widehat{\Delta})$ the dual of (A, Δ) , and by (D, Δ_D) the natural $*$ -algebraic quantum hypergroup structure on the space of fixed elements for the antipode squared \widehat{S}^2 on \widehat{A} . Finally, by $(\widehat{C}, \widehat{\Delta}_C)$ we denote the dual $*$ -algebraic quantum hypergroup of (C, Δ_C) .

Proposition 2.1. *There is a natural isomorphism of $*$ -algebraic quantum hypergroups $(\widehat{C}, \widehat{\Delta}_C) \rightarrow (D, \Delta_D)$.*

Proof. Denote by φ a non-zero positive left invariant functional for (A, Δ) . We recall from [3] that then $\varphi \circ S^2 = \varphi$, from which it immediately follows that $\varphi \circ E = \varphi$. Hence the restriction φ_C of φ to C is a non-zero positive left invariant functional for (C, Δ_C) .

We recall now from [20] and [5] that \widehat{A} (resp. \widehat{C}) can be identified with the space of functionals on A (resp. C) of the form $\varphi(\cdot a)$ with $a \in A$ (resp. $\varphi_C(\cdot c)$ with $c \in C$). Moreover, the map $a \in A \rightarrow \varphi(\cdot a) \in \widehat{A}$ is then a linear bijection (and similarly for C). Consider then the map

$$\Theta : D \rightarrow \widehat{C} : \varphi(\cdot a) \rightarrow \varphi_C(\cdot E(a)).$$

We claim that Θ is an isomorphism of $*$ -algebraic quantum groups.

In fact, since $\varphi \circ S^2 = \varphi$ and $\widehat{S}^2(\omega) = \omega \circ S^2$ for $\omega \in \widehat{A}$, it is immediately seen that $a \in C$ iff $\varphi(\cdot a) \in D$, so Θ is in any case a bijection. By the E -invariance of φ and the C -bimodularity of E , we have that Θ is also the restriction to D of the projection map

$$P : \widehat{A} \rightarrow C^\circ \supseteq \widehat{C} : \omega \rightarrow \omega|_C.$$

Denoting by \widehat{E} the equivalent of E for $(\widehat{A}, \widehat{\Delta})$, it is then also clear that $\langle \widehat{E}(\omega), a \rangle = \langle \omega, E(a) \rangle$ for $\omega \in \widehat{A}$ and $a \in A$.

Choose now $c, d, x \in C$ and write $\omega_1 = \varphi(\cdot c)$ and $\omega_2 = \varphi(\cdot d)$ as elements in $D \subseteq \widehat{A}$. Then

$$\begin{aligned} \langle \Theta(\omega_1 \cdot \omega_2), x \rangle &= \langle \omega_1 \cdot \omega_2, x \rangle \\ &= (\varphi \otimes \varphi)(\Delta(x)(c \otimes d)) \\ &= (\varphi \circ E \otimes \varphi \circ E)(\Delta(x)(c \otimes d)) \\ &= (\varphi \otimes \varphi)((E \otimes E)(\Delta(x))(c \otimes d)) \\ &= (\varphi_C \otimes \varphi_C)(\Delta_C(x)(c \otimes d)) \\ &= \langle \Theta(\omega_1) \cdot \Theta(\omega_2), x \rangle, \end{aligned}$$

proving that Θ is an algebra homomorphism. In the same way it can be shown to be $*$ -preserving.

Finally, choose $\omega_1, \omega_2 \in D$ and $c, d \in C$. Then

$$\begin{aligned} \langle (\Theta \otimes \Theta)(\Delta_D(\omega_1)(1 \otimes \omega_2)), c \otimes d \rangle &= \langle (\widehat{E} \otimes \widehat{E})(\widehat{\Delta}(\omega_1)(1 \otimes \omega_2)), c \otimes d \rangle \\ &= \langle \widehat{\Delta}(\omega_1)(1 \otimes \omega_2), c \otimes d \rangle \\ &= \langle \omega_1 \otimes \omega_2, (c \otimes 1)\Delta(d) \rangle \\ &= \langle \omega_1 \otimes \omega_2, (c \otimes 1)\Delta_C(d) \rangle \\ &= \langle \Theta(\omega_1) \otimes \Theta(\omega_2), (c \otimes 1)\Delta_C(d) \rangle \\ &= \langle \widehat{\Delta}_C(\Theta(\omega_1))(1 \otimes \Theta(\omega_2)), c \otimes d \rangle. \end{aligned}$$

This concludes the proof. □

3 $*$ -Algebraic quantum hypergroups associated with $SU_q(2)$

Let us now consider a concrete example, namely the compact quantum group $SU_q(2)$ ([21]) and its dual. We fix a number $0 < q < 1$.

We denote by $A = \text{Pol}(SU_q(2))$ the unital $*$ -algebra, generated by two elements a and b , satisfying the commutation relations

$$\begin{cases} a^*a + b^*b = 1 & ab = q \quad ba \\ aa^* + q^2bb^* = 1 & a^*b = q^{-1}ba^* \\ bb^* = b^*b. \end{cases}$$

It can be given a comultiplication Δ by the formulas

$$\Delta(a) = a \otimes a - qb^* \otimes b$$

$$\Delta(b) = b \otimes a + a^* \otimes b,$$

and it then becomes a $*$ -algebraic quantum group of compact type (i.e. a Hopf $*$ -algebra allowing a non-zero positive invariant functional).

The antipode S is determined by the formulas $S(a) = a^*$, $S(a^*) = a$, $S(b) = -qb$ and $S(b^*) = -q^{-1}b^*$. Since the set of elements

$$\{a^m b^k (b^*)^l, (a^*)^m b^k (b^*)^l \mid m, k, l \in \mathbb{N}\}$$

forms a vector space basis of A ([21]), we see that the space of fixed points of S^2 is the $*$ -algebra generated by a and a^* , and that the associated conditional expectation E is given by $E(a^m b^k (b^*)^l) = \delta_{k,l} a^m (b^*b)^k$ and $E((a^*)^m b^k (b^*)^l) = \delta_{k,l} (a^*)^m (b^*b)^k$.

Further keeping notation as in the previous section, it is then not difficult to see that the $*$ -algebra C can be identified with the unital $*$ -algebra generated by a single element c satisfying the commutation relation

$$(1 - cc^*) = q^2(1 - c^*c),$$

the identification with a sub- $*$ -algebra of A being given by sending c to a . As for the comultiplication $\Delta_C \cong (E \otimes E)\Delta_A$, one can easily calculate that

$$\begin{aligned} \Delta_C(1) &= 1 \otimes 1 \\ \Delta_C(c) &= c \otimes c \\ \Delta_C(c^*) &= c^* \otimes c^* \\ \Delta_C(c^*c) &= c^*c \otimes c^*c + q^2(1 - c^*c) \otimes (1 - c^*c). \end{aligned}$$

To know the full comultiplication, one should determine it on the vector space basis $(c^*)^k c^m$. We refrain from carrying out this computation here, since we will find an equivalent, but more natural formula for the comultiplication later on.

The dual $*$ -algebraic quantum group $(\widehat{A}, \widehat{\Delta})$ of $\text{Pol}(SU_q(2))$ will be of discrete type. The underlying $*$ -algebra is the (infinite) direct sum $*$ -algebra $\bigoplus_{l \in \frac{1}{2}\mathbb{N}} M_{2l+1}(\mathbb{C})$, where one designates the number l of some matrix block as the *spin* number. It can be shown ([19]) that there then exists a system of matrix units $e_{rs}^{(l)}$, where r and s take values in $\{-l, -l+1, \dots, l-1, l\}$, such that

$$\widehat{S}^2(e_{rs}^{(l)}) = q^{2(s-r)} e_{rs}^{(l)}.$$

It immediately follows that $D = \text{Fix}(\widehat{S}^2)$ is the commutative $*$ -algebra $\bigoplus_{l \in \frac{1}{2}\mathbb{N}} \mathbb{C}^{2l+1}$, and that \widehat{E} is determined by $\widehat{E}(e_{rs}^{(l)}) = \delta_{r,s} e_{rs}^{(l)}$. By duality, we then obtain that the comultiplication on C is cocommutative.

We note that also the center $\mathcal{Z} = \bigoplus_{l \in \frac{1}{2}\mathbb{N}} \mathbb{C}$ can be endowed with a quantum hypergroup structure, by giving it the comultiplication

$$\Delta_{\mathcal{Z}}(x) = (F' \otimes \iota)\Delta_D(x) = (\iota \otimes F)\Delta_D(x), \quad x \in D,$$

where $F(e_{rr}^{(l)}) = F'(e_{-r,-r}^{(l)}) = c_{(l)}q^{-2r}$, and $c_{(l)} = \frac{q^{-1}-q}{q^{-2n-1}-q^{2n+1}}$. This then determines a classical hypergroup structure on the set $\text{Irred}(SU_q(2))$. We note that this is a general result for all discrete *-algebraic quantum groups, and it has appeared (implicitly) in the study of Poisson boundaries of discrete quantum groups. Indeed, in this theory, one often first computes the Poisson boundary of this associated discrete quantum hypergroup, which is sometimes trivial (for example when the dual compact quantum group has commutative fusion rules, as is the case for $SU_q(2)$), but not always (for example, in [15] it is shown that for the universal discrete quantum groups $\widehat{A_u(F)}$, one obtains a hypergroup structure on the set $\mathbb{N} * \mathbb{N}$ with a non-trivial Poisson boundary).

Let us now examine the coproduct on the compact quantum hypergroup C more closely. We will in fact study the associated von Neumann algebraic picture. Indeed, one can embed $\text{Pol}(SU_q(2))$ as a σ -weakly dense sub-*-algebra in the von Neumann algebra $\mathcal{L}^\infty(SU_q(2)) := B(l^2(\mathbb{N})) \bar{\otimes} \mathcal{L}^\infty(\mathbb{Z})$, by the application

$$\begin{aligned} a &\rightarrow \left(\sum_{k \in \mathbb{N}_0} \sqrt{1 - q^{2k}} e_{k-1,k} \right) \otimes 1, \\ b &\rightarrow \left(\sum_{k \in \mathbb{N}} q^k e_{kk} \right) \otimes S, \end{aligned}$$

where e_{kl} are the natural matrix units of $B(l^2(\mathbb{N}))$, and S denotes the forward shift. Then Δ can be extended to a normal unital *-homomorphism $\Delta : \mathcal{L}^\infty(SU_q(2)) \rightarrow \mathcal{L}^\infty(SU_q(2)) \bar{\otimes} \mathcal{L}^\infty(SU_q(2))$, making $(\mathcal{L}^\infty(SU_q(2)), \Delta)$ into a von Neumann algebraic quantum group.

We note then that E can also be extended to a normal conditional expectation

$$E : \mathcal{L}^\infty(SU_q(2)) \rightarrow B(l^2(\mathbb{N})) \subseteq \mathcal{L}^\infty(SU_q(2)) : e_{kl} \otimes S^m \rightarrow \delta_{m,0} e_{kl}.$$

In this way, we can also extend Δ_C to a normal, unital, *-preserving coassociative comultiplication

$$\Delta_C : B(l^2(\mathbb{N})) \rightarrow B(l^2(\mathbb{N})) \bar{\otimes} B(l^2(\mathbb{N})) : x \rightarrow (E \otimes E)(\Delta(x)).$$

We have then endowed $B(l^2(\mathbb{N}))$ with the structure of a ‘(compact) von Neumann algebraic quantum hypergroup’ structure, although such a concept has as of yet not been defined rigourously. Note that by duality, the predual $B(l^2(\mathbb{N}))_*$ then becomes ‘a completely contractive Banach *-algebra’, where the *-structure is implemented by the unitary antipode, so that $\omega_{rs}^* = \omega_{sr}$, with ω_{rs} denoting the functional

$$\omega_{rs}(e_{kl}) = \delta_{r,k} \delta_{s,l}$$

in $B(l^2(\mathbb{N}))_*$. Since Δ_C is cocommutative, $B(l^2(\mathbb{N}))_*$ will be commutative.

Let us now compute the comultiplication on the basis elements e_{rs} . We use the explicit implementation of Δ by a unitary v as obtained in [11], Theorem 4.1. One then easily arrives at the expression

$$\Delta_C(e_{rs}) = \sum_{\substack{k,l \\ k,l \geq s-r}}^{\infty} f(\min\{k,l\}, |k-l|, s) f(\min\{k,l\} + r - s, |k-l|, r) e_{k+r-s,k} \otimes e_{l+r-s,l},$$

where f is the function on \mathbb{N}^3 determined by

$$f(k, n, r) = \frac{(q^{2+2n}; q^2)_\infty^{1/2} (q^{2+2n}; q^2)_k^{1/2}}{(q^2; q^2)_k} (-q^{n+1})^{r-k} p_k(q^{2r}; q^{2n}, 0 \mid q^2)$$

with $p_k(x; q^{2n}, 0 \mid q^2) = {}_2\phi_1(q^{-2k}, 0; q^{2n+2}; q^2, q^2 x)$ (the so-called *Wall polynomials*). Using formula (4.7) of [11], we see that the expression for $\Delta_C(e_{00})$ can be simplified to

$$\Delta_C(e_{00}) = (q^2; q^2)_\infty \sum_{k,l=0}^{\infty} \frac{q^{2kl}}{(q^2; q^2)_k (q^2; q^2)_l} e_{kk} \otimes e_{ll}.$$

Note that it immediately follows from these computations that Δ_C restricts to the von Neumann algebra $\mathcal{L}^\infty(\mathbb{N}) \subseteq B(l^2(\mathbb{N}))$ of diagonal operators:

$$\Delta_C(\mathcal{L}^\infty(\mathbb{N})) \subseteq \mathcal{L}^\infty(\mathbb{N}) \bar{\otimes} \mathcal{L}^\infty(\mathbb{N}).$$

But this restriction will give the double coset quantum group structure on $S^1 \backslash SU_q(2) / S^1$ as studied in [1]. Indeed, with

$$F : \mathcal{L}^\infty(SU_q(2)) \rightarrow \mathcal{L}^\infty(S^1 \backslash SU_q(2) / S^1) = W^*(b^*b) : e_{rs} \otimes S^n \rightarrow \delta_{r,s} \delta_{n,0} e_{rr} \otimes 1$$

the natural conditional expectation obtained by integrating out the action α of $S^1 \times S^1$ on $\mathcal{L}^\infty(SU_q(2))$ determined by

$$\begin{aligned} \alpha_{(\varphi, \theta)}(a) &= e^{i(\varphi + \theta)} a, \\ \alpha_{(\varphi, \theta)}(b) &= e^{i(\varphi - \theta)} b, \end{aligned}$$

we have that $F \circ E = F$, so that Δ_C , restricted to $\mathcal{L}^\infty(S^1 \backslash SU_q(2) / S^1)$, gives the natural comultiplication $\Delta_{\mathcal{L}^\infty(S^1 \backslash SU_q(2) / S^1)} = (F \otimes F)\Delta$.

Finally, let us look at the bicrossed product construction of the first section in this case. We have now, using again the notations of that section, that $T = G = \{q^{2n} \mid n \in \mathbb{Z}\}$, and so $\widehat{G} \cong S^1 \cong \mathbb{R} / \frac{\pi}{\ln(q)} \mathbb{Z}$. The bicrossed product B of A and $K(G)$ will then be a *-algebraic quantum group of non-discrete, non-compact type. It can be implemented on the Hilbert space $l^2(\mathbb{N}) \otimes l^2(\mathbb{Z})$ by letting A act as in the previous way, and $K(G)$ by

$$\delta_{q^{2r}} \rightarrow 1 \otimes \delta_r.$$

Its left invariant positive functional can be implemented by the nsf weight $\theta \otimes \text{Tr}$, where Tr is the canonical trace on $B(l^2(\mathbb{Z}))$, and θ is the state on $B(l^2(\mathbb{N}))$ determined by $\theta(e_{rs}) = \delta_{r,s} (1 - q^2) q^{2r}$. It also follows from this that the von Neumann algebraic quantum group associated to B has $B(l^2(\mathbb{N})) \bar{\otimes} B(l^2(\mathbb{Z}))$ as its associated von Neumann algebra. This provides yet another example of a von Neumann algebraic quantum group whose underlying von Neumann algebra is an (infinite-dimensional) type I factor. However, it seems to be the first example of such a von Neumann algebraic quantum group which is *regular* (which follows since *any* *-algebraic quantum group has a regular associated von Neumann algebraic quantum group).

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